2026 SMC Second Round Scheme

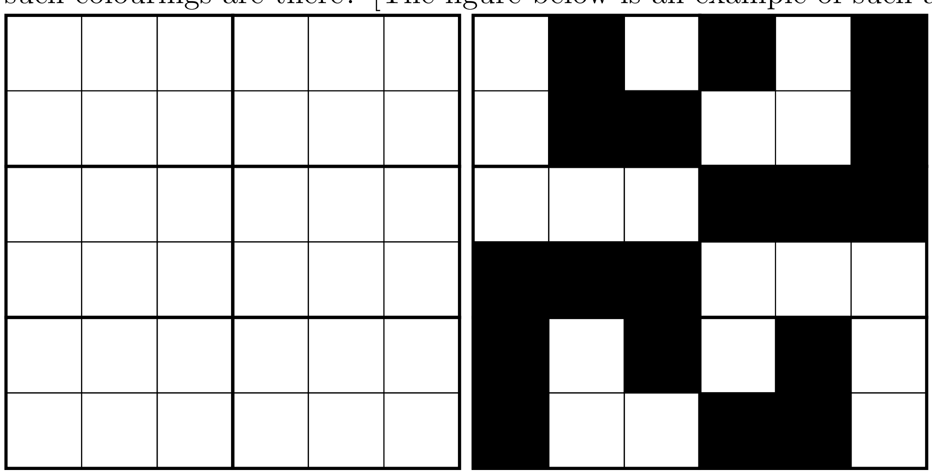
28th February 2026

1 PROBLEMS

Problem 1: Let $a_1, a_2, a_3, \cdots,$ be an infinite sequence of integers such that $a_{n+1} = a_n + 2n$, for all $n \in \mathbb{N}$. Show that a_n is not prime for some $n \in \mathbb{N}$.

Problem 2: Let $ABCD$ be a quadrilateral with diagonals of equal lengths intersecting at E. Suppose the perpendicular bisectors of AB and CD intersect at F, and the perpendicular bisectors of BC and DA intersect at G. Show that EF is perpendicular to EG.

Problem 3: Given an (6×6) grid, we partition it into 6 grids of 2×3 each by stacking them 2 times horizontally and 3 times vertically. A colouring is called *Special Sudoku* if each unit square is coloured white or black such that each colour appears exactly 3 times in each row, column and each of the smaller 2×3 grids. How many such colourings are there? [The figure below is an example of such a colouring]

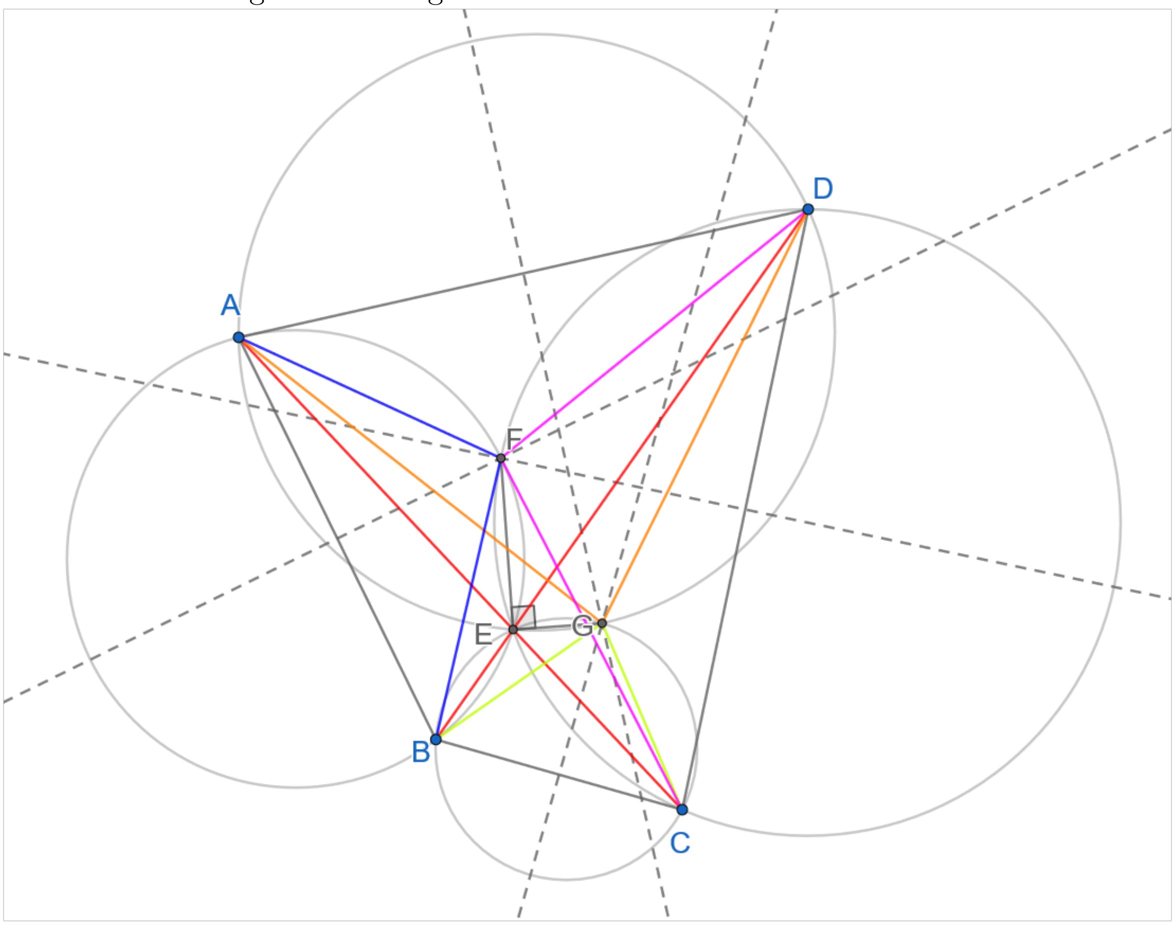


Problem 4: Find all functions $f : \mathbb{N} \longrightarrow \mathbb{N}$ satisfying $f(f(x)) + f(x) = 6x, \forall x \in \mathbb{N}$.

2 SOLUTIONS

Problem 1: Have $a_1 = a_1, a_2 = a_1 + 2, a_3 = a_1 + 2 + 4, a_4 = a_1 + 2 + 4 + 6, \cdots$. Inductively, get $a_n = a_1 + n(n-1)$.
If a_1 is not prime then we are done.
If $a_1 = p$ is a prime. Then a_n can be factorized if $n = a_1$ or $n - 1 = a_1$. Thus take $n = p$ and then $a_p = p^2$ is not prime.

Problem 2: Using directed angles.



Claim 1:$\triangle ACF \equiv \triangle BDF$ (and $\triangle ACG \equiv \triangle BDG$).
Indeed, observe that $AF = BF, CF = DF, AC = BD$.
Claim 2:$ABEF$ is cyclic (same for $CDEF, BCEG, ADEG$).
Indeed, $\angle EAF = \angle CAF = \angle DBF = \angle EBF$.
Claim 3:$\triangle ABF \sim \triangle CDF$ (and $\triangle ADG \equiv \triangle CBG$).
First observe that they are all isosceles. It suffices to show $\angle AFB = \angle CFD$. Indeed, $\angle AFB = \angle AFC + \angle CFB = \angle BFD + \angle CFB = \angle CFB + \angle BFD = \angle CFD$.
Claim 4:$\angle AEF = \angle FED$ (and $\angle AEG = \angle GEB$).
Indeed, $\angle AEF = \angle ABF = \angle FAB = \angle FCD = \angle FED$.

Finally, $\angle FEG = \angle FEA + \angle AEG = \angle DEF + \angle GEB = \angle BEF + \angle GEB = \angle GEB + \angle BEF = \angle GEF = -\angle FEG$. Thus either $\angle FEG = 90°$ and we are done, or $\angle FEG = 0°$ which means FEG collinear.

Problem 3: Consider two kinds of 2×3 grid acceptable colouring: 3 black on are row (hereby referred to as the *row type*) or 2 black in a column (hereby referred to as the *column type*), or else have a *mixed type*. We call a 2×3 grid *upper* (U) if the top row has more black than white, and call it *lower* (L) otherwise. Let the *signature* of a row or column type be the triple (a, b, c), where the first column has a blacks, the second column has b blacks and third column has c blacks. This is a permutation of $(0, 1, 2)$ for column types and $(1, 1, 1)$ for row type. Let the *signature* of a mixed type the position of be the triple (a, b, c), where the first grid has a blacks, the second grid has b blacks and third grid has c blacks along the row containing only one black. This is a permutation of $(1, 0, 0)$.

First, observe that it is not possible to have two different types of 2×3 along same larger rows. Moreover, this must comprise of an upper and lower pair. Thus, we shall label to each larger row according to the its 2×3 grid type: R for row type, C for the column type, M for mixed type.

Next, observe that a column type is uniquely determined by its signature independently combined with whether or not it is upper or lower.

Now consider 4 kinds of collection of labels: RRR, RR[.], R[.][.] and [.][.][.]. [.] is a mix of M and C on each larger column.

Case 1 (RRR): There are $2 \times 2 \times 2 = 8$ in total since their signature is unique so that only need to consider the UL and LU ordering (independently) for each larger row.

Case 2 (RR[.]): There are $3 \times 3 \times 3 \times 8 = 27 \times 8$ in total. The first 3 is for choosing the position of the [.]=M. The other 3s are for choosing the signature of M's on either side. The 8 is same as in case 1.

Case 3 (R[.][.]): There are $3 \times [3! + 3^2] \times [3! + 3^2] \times 8 = 675 \times 8$ in total. The 3 is for choosing the position of the R. The 3! is for choosing the signature of two C's. The 3^2 is for choosing the signature of two M's. The 8 is same as in case 1.

Case 4 ([.][.][.]):There are $[3! \times 2 \times 1 + 3 \times 3 \times 3! + 3^3] \times [3! \times 2 \times 1 + 3 \times 3 \times 3! + 3^3] \times 8 = 93 \times 93 \times 8$ in total. CCC: The 3! is for choosing the signature of first C, the 2 is for choosing the signature of second C and the 1 is for choosing the signature of last C (the unique signature such that the sum of all signatures is $(3, 3, 3)$), all within the same larger column. MCC: $3 \times 3 \times 3!$. MMM: 3^3 . The 8 is same as in case 1.

Thus have $9352 \times 8 = 74816$.

Problem 4: Fix x. Then have a sequence $a_n = f^n(x), n \geq 0$. Consequently, $a_{n+2} + a_{n+1} = 6a_n, n \geq 0$. This has characteristic equation $t^2 + t - 6 = 0$. This has roots $t = 2, -3$. So, $a_n = \alpha 2^n + \beta(-3)^n$. Have $x = a_0 = \alpha + \beta, f(x) = a_1 = 2\alpha - 3\beta$. Therefore, $5f^n(x) = 5a_n = [3x + f(x)]2^n + [2x - f(x)](-3)^n$. In particular, when n is even, have $5f^n(x)3^{-n} = [2x - f(x)] + (2/3)^n[3x + f(x)] > 0$, so $f(x) \leq 2x$ else must have $f(x) > 2x$ and can find n large enough such that $(2/3)^n < \frac{f(x)-2x}{3x+f(x)}$. Now, $f(f(x)) \leq 2f(x), f(x) \leq 2x \implies f(f(x)) + f(x) \leq 6x$ and thus equality implies $f(x) = 2x$.

3 MARKING SCHEME

Problem 1:

- Considering the case a_1 not prime or a_1 not positive (proof by contradiction automatically earns this point) [**1 Point**].
- Simplifying a_n in terms of a_1 [**2 Points**].
- Completing the proof [**4 Points**].
  - Testing for few examples [$\leq$**2 Points**].

Problem 2:

- Claim 1 [**2 Points**].
  - Identifying only 1 pair [**1 Point**].
- Claim 2 [**2 Points**].
  - Identifying only 1 or 2 circles [**1 Point**].
- Claim 3 [**1 Point**].
- Claim 4 [**1 Point**].
- Completing the proof [**1 Point**].

Problem 3:

- Case 1 [**1 Point**].
- Case 2 [**1 Points**].

- Case 3 [**2 Points**].
- Case 3 [**3 Points**].

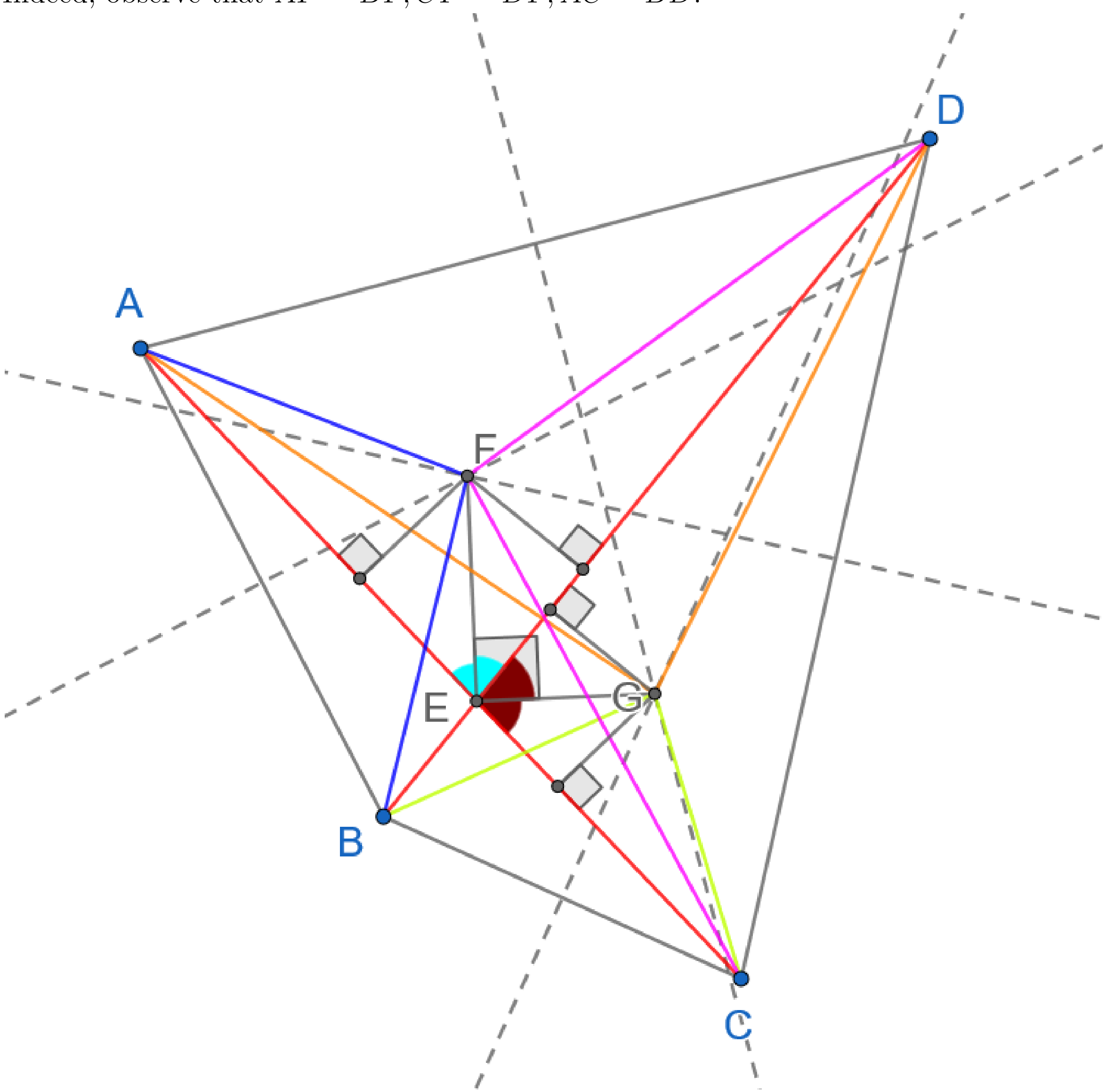
Problem 4:

- Observing or making the claim that if $f(x) \neq x$, then $f^n(x)$ will eventually be negative [**3 Points**].
  - Finding $f(1) = 2$ [**1 Point**].
  - Finding $f(2) = 4$ or $f(k) = 2k$ some $k > 1$ [**1 Point**].
- Proving the claim that if $f(x) \neq x$, then $f^n(x)$ will eventually be negative [**4 Points**].
  - Expressing $f^n(x)$ in terms of x and $f(x)$ [**2 Point2**].
  - Showing $f(x) \geq 2x$ or $f(x) \leq 2x$ [**1 Point**].

4 EXTRA SOLUTIONS DUE TO CONTESTANTS

4.1 Solution to Problem 2 due to Ojomu Akeem Adewale and Sampson Abasiofon Otobong

Problem 2: Getting the congruent triangles as in the official solution.
Claim 1: $\triangle ACF \equiv \triangle BDF$ (and $\triangle ACG \equiv \triangle BDG$).
Indeed, observe that $AF = BF, CF = DF, AC = BD$.



Claim 2: G lies on the angle bisector of $\angle AEB$ ($\angle CED$).
Consider the altitudes from G in congruent triangles $\triangle BDG$ and $\triangle ACG$. Due to the congruency, these altitudes are equal, hence G is equidistant from BD and AC, but this is part of the locus of points equidistant from BD and AC, which is the angle bisector in the claim, thus G lies on the angle bisector.

Claim 3: F lies on the angle bisector of $\angle AED$ ($\angle BEC$).
Consider the altitudes from F in congruent triangles $\triangle BDF$ and $\triangle ACF$. Due to the congruency, these altitudes are equal, hence F is equidistant from BD and AC, but this is part of the locus of points equidistant from BD and AC, which is the angle bisector in the claim, thus F lies on the angle bisector.
From this, we have that $\angle FEG = \angle FED + \angle DEG = \frac{1}{2}(\angle AED + \angle DEC) = \frac{1}{2}(180°) = 90°$.