SMC Round 2, 2022

Problems, Solutions and Marking Scheme

By Onah Pius Aje
For SMA
December 13, 2022

1 Problems

Problem 1:

Let $n \in \mathbb{N}$. Prove that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}+\sqrt{y_i}} \geq \frac{\sqrt{n}}{\sqrt{\sum_{i=1}^{n} x_i}+\sqrt{\sum_{i=1}^{n} y_i}}$$

$\forall x_i, y_i \in \mathbb{R}^+, i = 1, 2, \cdots, n$.

Problem 2:

Find all triples of prime numbers (p, q, r), such that $q|r-1$, and

$$\frac{r\left(p^{q-1}-1\right)}{q^{p-1}-1}$$

is prime.

Problem 3:

The numbers $2, 3, 4, \cdots, 100$ are written on a board. Chibuike and Ismail play a game of erasing numbers from the board using the following rule: If the number, a, is erased, then only numbers, b, such that $\gcd(a, b) = 1$, can be erased in subsequent turns.
The game ends when no such b exists, to be erased, and the person that erased last wins.

If Chibuike starts the game, does there exist a winning strategy for him? (Determine with proof.)

Problem 4:
Let Ω be a circle with center O with P, a point lying outside. Tangents from P are drawn to touch the circle at A and B. A point, T is arbitrary chosen on major arc AB, and D is the foot of T on AB. K, L, M, N are the mid points of TA, TB, TD, AB respectively. PT intersects MN at point S. Lines l_a and l_b are the reflection of OA and OB over the angle bisectors of $\angle SAL$ and $\angle SBK$, respectively. Show that l_a, l_b and TD are concurrent.

2 Solutions

Problem 1:

Let $n \in \mathbb{N}$. Prove that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}+\sqrt{y_i}} \geq \frac{\sqrt{n}}{\sqrt{\sum_{i=1}^{n} x_i}+\sqrt{\sum_{i=1}^{n} y_i}}$$

$\forall x_i, y_i \in \mathbb{R}^+, i = 1, 2, \cdots, n$.

Solution 1

Let $AM(a_i), HM(a_i), QM(a_i)$ denote the arithmetic mean, the harmonic mean, and the quadratic mean respectively, of $a_1, a_2, \cdots, a_n \in \mathbb{R}^+$. Then it follows that $QM(a_i) \geq AM(a_i) \geq HM(a_i) > 0$.

Thus,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}+\sqrt{y_i}} = AM\left(\frac{1}{\sqrt{x_i}+\sqrt{y_i}}\right)$$

$$\geq HM\left(\frac{1}{\sqrt{x_i}+\sqrt{y_i}}\right) = \frac{1}{AM\left(\sqrt{x_i}+\sqrt{y_i}\right)} = \frac{1}{AM\left(\sqrt{x_i}\right)+AM\left(\sqrt{y_i}\right)}$$

$$\geq \frac{1}{QM\left(\sqrt{x_i}\right)+QM\left(\sqrt{y_i}\right)} = \frac{1}{\sqrt{AM\left(x_i\right)}+\sqrt{AM\left(y_i\right)}} = \frac{\sqrt{n}}{\sqrt{\sum_{i=1}^{n} x_i}+\sqrt{\sum_{i=1}^{n} y_i}}.$$

Solution 2 (Induction)

CLAIM: Let $\alpha \in [0, 1]$ and let $x, y, X, Y \in \mathbb{R}^+$. Then

$$\frac{\alpha}{\sqrt{X}+\sqrt{Y}} + \frac{1-\alpha}{\sqrt{x}+\sqrt{y}} \geq \frac{1}{\sqrt{\alpha X+(1-\alpha)x}+\sqrt{\alpha Y+(1-\alpha)y}}.$$

Proof 1 of claim:
Fix $x, y, X, Y \in \mathbb{R}^+$ and consider the function in α:

$$F(\alpha) = f(\alpha) - \frac{1}{g(\alpha)},$$

where

$$f(\alpha) = \left(\frac{\alpha}{\sqrt{X}+\sqrt{Y}} + \frac{1-\alpha}{\sqrt{x}+\sqrt{y}}\right), g(\alpha) = \sqrt{\alpha X+(1-\alpha)x}+\sqrt{\alpha Y+(1-\alpha)y}.$$

Observe that $F(0) = F(1) = 0$, so it is enough to show that F is concave.
First observe that f is linear and g is concave since

$$g''(\alpha) = -\frac{(X-x)^2}{4\sqrt{\alpha X+(1-\alpha)x}^3} - \frac{(Y-y)^2}{4\sqrt{\alpha Y+(1-\alpha)y}^3} \leq 0.$$

Indeed,

$$F'(\alpha) = f'(\alpha) + \frac{g'(\alpha)}{g(\alpha)^2} \implies F''(\alpha) = 0 + \frac{g''(\alpha)}{g(\alpha)^2} - \frac{2g'(\alpha)^2}{g(\alpha)^3} \leq 0. \quad \square$$

Proof 2 of claim:
Let $h : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be given by

$$h(x, y) := \frac{1}{\sqrt{x} + \sqrt{y}}.$$

This can be viewed as a surface in 3D.
Then the claim can be written as

$$\alpha h(X, Y) + (1 - \alpha)h(x, y) \geq h\left[\alpha X + (1 - \alpha)x, \alpha Y + (1 - \alpha]y\right) = h\left[\alpha(X, Y) + (1 - \alpha)(x, y)\right].$$

That is, h is convex by Jensen's inequality. This equivalent to checking that the Hessian of h is positive definite.
The Hessian is given by

$$H(h) = \begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix}.$$

I leave the rest of the working to the reader. $\square$

Now, we are ready to proceed via induction on n.

- $n = 1$: This is trivial as we have equality.
- $n = 2$: RTP:

$$\frac{1}{2}\left(\frac{1}{\sqrt{x_1} + \sqrt{y_1}} + \frac{1}{\sqrt{x_2} + \sqrt{y_2}}\right) \geq \frac{\sqrt{2}}{\sqrt{x_1 + x_2} + \sqrt{y_1 + y_2}}.$$

  To see this, take $\alpha = \frac{1}{2}, X = x_1, Y = y_1, x = x_2, y = y_2$ in **claim**.

- Assume true for $n = k$, some $k \geq 2$:

$$S = \frac{1}{k}\sum_{i=1}^{k} \frac{1}{\sqrt{x_i} + \sqrt{y_i}} \geq \frac{\sqrt{k}}{\sqrt{\sum_{i=1}^{k} x_i} + \sqrt{\sum_{i=1}^{k} y_i}}$$

- Show for $n = k + 1$: RTP

$$\frac{1}{k+1}\sum_{i=1}^{k+1} \frac{1}{\sqrt{x_i} + \sqrt{y_i}} \geq \frac{\sqrt{k+1}}{\sqrt{\sum_{i=1}^{k+1} x_i} + \sqrt{\sum_{i=1}^{k+1} y_i}}.$$

  Now let $x = x_{k+1}, y = y_{k+1}$, and

$$X = \frac{1}{k}\sum_{i=1}^{k} x_i, Y = \frac{1}{k}\sum_{i=1}^{k} y_i.$$

  So,

$$LHS = \frac{1}{k+1}\sum_{i=1}^{k+1} \frac{1}{\sqrt{x_i} + \sqrt{y_i}} = \frac{1}{k+1}\left(kS + \frac{1}{\sqrt{x} + \sqrt{y}}\right) \geq \frac{1}{k+1}\left(\frac{k}{\sqrt{X} + \sqrt{Y}} + \frac{1}{\sqrt{x} + \sqrt{y}}\right),$$

$$\geq \frac{\sqrt{k+1}}{\sqrt{kX+x}+\sqrt{kY+y}} = RHS.$$

The first inequality follows from assume step, while the last inequality follows by taking $\alpha = \frac{k}{k+1}$ in **claim**. □

Solution 3 (Partly induction)

CLAIM: Let $x, y, X, Y \in \mathbb{R}^+$. Then

$$\frac{1}{\sqrt{X}+\sqrt{Y}} + \frac{1}{\sqrt{x}+\sqrt{y}} \geq \frac{2\sqrt{2}}{\sqrt{X+x}+\sqrt{Y+y}}.$$

Proof 1 of claim:
Apply **claim** as in **solution 2**. □

Proof 2 of claim:
Let $X = a^2, Y = b^2, x = c^2, y = d^2$. Then **claim** becomes

$$\frac{1}{a+b} + \frac{1}{c+d} \geq \frac{2\sqrt{2}}{\sqrt{a^2+c^2}+\sqrt{b^2+d^2}}$$

$$\iff \sqrt{a^2+c^2}+\sqrt{b^2+d^2} \geq \frac{2\sqrt{2}(a+b)(c+d)}{a+b+c+d}.$$

Indeed,

$$\sqrt{a^2+c^2}+\sqrt{b^2+d^2} \geq \frac{a+c+b+d}{\sqrt{2}} \geq \frac{2\sqrt{2}(a+b)(c+d)}{a+b+c+d},$$

The first in equality follows from QM-AM, while the second follows from AM-GM (after cross multiplication) or AM-HM (indirectly). □

First, we prove the problem for $n = 2^m$, some $m \in \mathbb{N}_0$, by induction on m.

- $m = 0$: This is trivial as we have equality.
- $m = 1$: RTP:

$$\frac{1}{2}\left(\frac{1}{\sqrt{x_1}+\sqrt{y_1}} + \frac{1}{\sqrt{x_2}+\sqrt{y_2}}\right) \geq \frac{\sqrt{2}}{\sqrt{x_1+x_2}+\sqrt{y_1+y_2}}.$$

  This is exactly the same as **claim**.
- Assume true for $m = k$, some $k \geq 1$:

$$\frac{1}{2^k}\sum_{i=1}^{2^k} \frac{1}{\sqrt{x_i}+\sqrt{y_i}} \geq \frac{\sqrt{2^k}}{\sqrt{\sum_{i=1}^{2^k} x_i}+\sqrt{\sum_{i=1}^{2^k} y_i}}$$

- Show for $m = k+1$: RTP

$$\frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}} \frac{1}{\sqrt{x_i}+\sqrt{y_i}} \geq \frac{\sqrt{2^{k+1}}}{\sqrt{\sum_{i=1}^{2^{k+1}} x_i}+\sqrt{\sum_{i=1}^{2^{k+1}} y_i}}.$$

Indeed,

$$\frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}}\frac{1}{\sqrt{x_i}+\sqrt{y_i}}=\frac{1}{2}\left\{\frac{1}{2^k}\sum_{i=1}^{2^k}\frac{1}{\sqrt{x_i}+\sqrt{y_i}}+\frac{1}{2^k}\sum_{i=1}^{2^k}\frac{1}{\sqrt{x_{i+2^k}}+\sqrt{y_{i+2^k}}}\right\}$$

$$\geq \frac{1}{2}\left\{\frac{\sqrt{2^k}}{\sqrt{\sum_{i=1}^{2^k}x_i}+\sqrt{\sum_{i=1}^{2^k}y_i}}+\frac{\sqrt{2^k}}{\sqrt{\sum_{i=1}^{2^k}x_{i+2^k}}+\sqrt{\sum_{i=1}^{2^k}y_{i+2^k}}}\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{\sqrt{\frac{\sum_{i=1}^{2^k}x_i}{2^k}}+\sqrt{\frac{\sum_{i=1}^{2^k}y_i}{2^k}}}+\frac{1}{\sqrt{\frac{\sum_{i=1}^{2^k}x_{i+2^k}}{2^k}}+\sqrt{\frac{\sum_{i=1}^{2^k}y_{i+2^k}}{2^k}}}\right\}$$

$$\geq \frac{\sqrt{2}}{\sqrt{\frac{\sum_{i=1}^{2^k}x_i}{2^k}+\frac{\sum_{i=1}^{2^k}x_{i+2^k}}{2^k}}+\sqrt{\frac{\sum_{i=1}^{2^k}y_i}{2^k}+\frac{\sum_{i=1}^{2^k}y_{i+2^k}}{2^k}}}$$

$$=\frac{\sqrt{2^{k+1}}}{\sqrt{\sum_{i=1}^{2^{k+1}}x_i}+\sqrt{\sum_{i=1}^{2^{k+1}}y_i}}.$$

The first inequality follows from assume step, while the last inequality follows from **claim**.

Next, we prove for general $n \neq 2^m$. We may assume $n < 2^m$, some $m > 1$. More specifically, we shall show that if the problem holds for some $n = k > 3$, then it also holds for $n = k-1$. This is a finite downward induction.

The construction is quite straight forward. To prove for $n = k-1$, apply the case $n = k$ by taking x_i for $i = 1, 2, \cdots k-1$ as before, then take x_k as their arithmetic mean (same goes for y_i). The rest is easy deduction. $\square$

Problem 2:

Find all triples of prime numbers (p, q, r), such that $q|r-1$, and

$$\frac{r\left(p^{q-1}-1\right)}{q^{p-1}-1}$$

is prime.

Solution

Let

$$\frac{r\left(p^{q-1}-1\right)}{q^{p-1}-1} = s,$$

then

$$r\left(p^{q-1}-1\right) = s\left(q^{p-1}-1\right).$$

Note that

$$r = s \iff p^{q-1}-1 = q^{p-1}-1 \iff p^{q-1} = q^{p-1} \iff p^{\frac{1}{p-1}} = q^{\frac{1}{q-1}} \iff p = q.$$

To see the final step, take ln and consider the function $f(x) = \frac{\ln(x)}{x-1}$. Then $f'(x) = \frac{x-1-x\ln(x)}{x(x-1)^2} < 0$ for $x > 1$, since $x\ln(x) + 1 - x = \int_1^x \ln(t)\ \mathrm{d}t$.

In this case, need $q|r-1$. Thus we get solution (q, q, r), where r is an odd prime and q is a prime divisor of $r-1$.

Suppose $r \neq s$, then $p \neq q$. Thus by FLT, have

$$p|\left(q^{p-1}-1\right) \implies p|r\left(p^{q-1}-1\right) \implies p|r,$$

$$q|\left(p^{q-1}-1\right) \implies q|s\left(q^{p-1}-1\right) \implies q|s.$$

Therefore, $p = r, q = s$.
Need $p^q - p = q^p - q$ and $q|p-1$.
Consider the function $g(x) = q^x - q - x^q + x$.
Have $g(q) = 0$ and $g'(x) = q^x\ln(q) - qx^{q-1} + 1 > q[q^{x-1} - x^{q-1}] > 0$ for $x > q \geq 3$. That is, g is strictly increasing (and hence positive) in the interval $(q, +\infty)$, when $q \geq 3$.
However, $g(p) = 0, p > q$. Thus, $q = 2$.

Now for $q = 2$, have $g(4) = 2$ and $g'(x) = 2^x\ln(2) - 2x + 1 > 2^{x-1} - 2x = \int_4^x [2^{t-1}\ln(2) - 2]\ \mathrm{d}t \geq 0$ for $x \geq 4$. That is, g is strictly increasing (and hence positive) in the interval $[4, +\infty)$.
Again, $g(p) = 0, p \geq 3$, so we conclude that $p = 3$.
Therefore, $p = r = 3, q = 2$.

Note 1:
The first inequality may be avoided by taking the two cases: $p = q$ and $p \neq q$, as opposed to taking cases $r = s$ and $r \neq s$ as in the proof above.

Note 2:
Since the domains can be restricted to positive integers, the above inequalities can also be shown using **induction** or **standard** inequality. The induction approach is standard, so I will only present the standard inequality application in the case of $p^q - p = q^p - q, q|p-1$:

If $q \geq 3$.
Then $p^q > p^q - p = q^p - q > q^{p-1}$. Thus $p > q^{\frac{p-1}{q}}$. (This still holds in the case of $p^{q-1} = q^{p-1}$.)
$q|p-1 \implies p = 2qk+1$, for some $k \geq 1$.
So $2kq + 1 = p > q^{2k} \implies 2k \geq q^{2k-1} \geq 1 + (q-1)\left(q^{2k-2} + q^{2k-2} + \cdots + q + 1\right) \geq 1 + 2(2k-1)$.
This is a contradiction.

If $q = 2$. (Then $p \geq 3$.)
First, $n \geq 1 \implies 2^n = 2\left(1 + 1 + 2 + 4 + \cdots + 2^{n-3} + 2^{n-2}\right) \geq 2n$.
Hence,

$$p\left(\frac{p-1}{2}\right) = 2^{p-1} - 1 = \left(2^{\frac{p-1}{2}} + 1\right)\left(2^{\frac{p-1}{2}} - 1\right) \geq \left(2\left(\frac{p-1}{2}\right) + 1\right)\left(2\left(\frac{p-1}{2}\right) - 1\right) = p(p-2).$$

So, $p = 3$.

Problem 3:

The numbers $2, 3, 4, \cdots, 100$ are written on a board. Chibuike and Ismail play a game of erasing numbers from the board using the following rule: If the number, a, is erased, then only numbers, b, such that $\gcd(a, b) = 1$, can be erased in subsequent turns.
The game ends when no such b exists, to be erased, and the person that erased last wins.

If Chibuike starts the game, does there exist a winning strategy for him? (Determine with proof.)

Solution

Chibuike (the first player) has a wining strategy as follows:

First note that every pair of numbers erased will be coprime. Let S denote the set of these numbers and let p denote a prime. This implies that

- If $a \in S$ and $p|a$, then $p < 100$. There are exactly 25 such primes.
- If $p < 100$, then there exists exactly one $a \in S$ such that $p|a$.
- If $a \in S$, then there are at most 3 primes satisfying $p|a$. Moreover, they form one of the sets $\{2, 3, 11\}, \{2, 3, 13\}$, or a subset of $\{2, 3, 5, 7\}$. Thus only one such number can belong to S.
- If $a \in S$ has exactly 2 prime divisors, then each must have a divisor from the set $\{2, 3, 5, 7\}$. Thus, there are at most 4 such numbers.
- If $a \in S$ has 3 prime divisors, then there is at most one other element of S with more than one prime divisor, except in the special cases when $S = \{66, 91, 85, \cdots\}, \{66, 91, 85, \cdots\}, \{66, 91, 95, \cdots\}, \{78, 77, 85, \cdots\}$, or $\{78, 77, 95, \cdots\}$.
- If $p > 50$, then $p \in S$.

In what follows, we present a simple strategy:
To begin the game, Chibuike erases $14 = 2 \cdot 7$.

- If Ismail erases 3^a, then Chibuike will erase 55.
- If Ismail erases 5^a, then Chibuike will erase 33.
- If Ismail erases $3^a 5^b$, then Chibuike will erase 11.
- If Ismail erases $3^a p$, where $p > 10$ is prime, then Chibuike will erase 5.
- If Ismail erases $5p$, where $p > 10$ is prime, then Chibuike will erase 3.
- If Ismail erases $p > 10$, the Chibuike will erase 15.

After this, no player can erase a number that is a multiple of $2, 3, 5$ or 7, so it must be prime greater than 10.

Since 5 primes have been used up so far, the remaining 20 primes ensures that Chibuike erases the last prime.

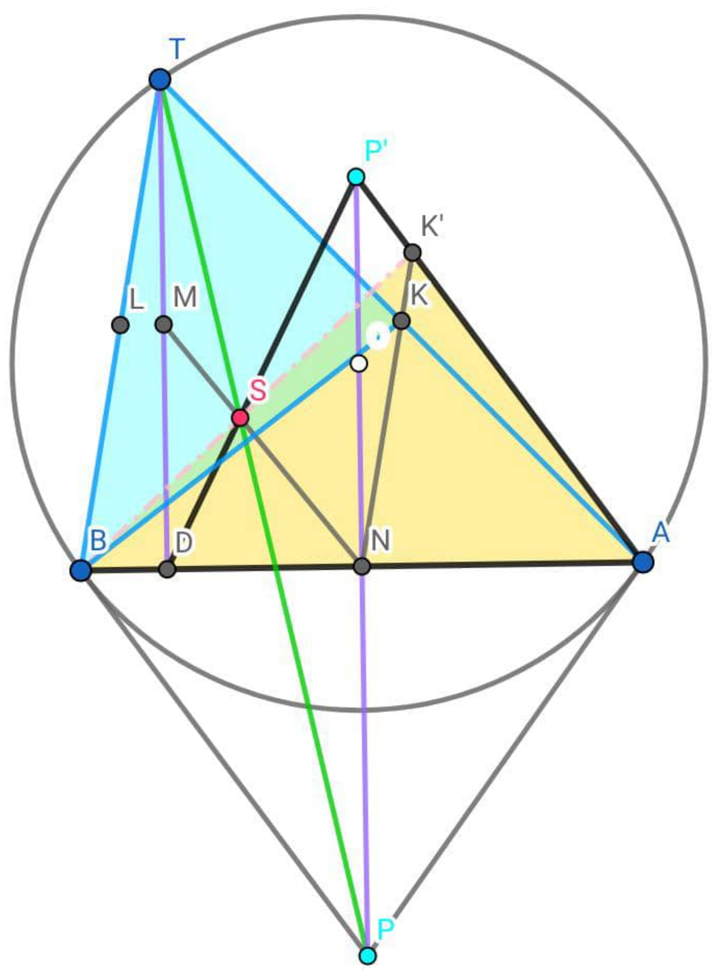
Note that in the strategy above, the role of 2 and 3 can be swapped. Same goes for 5 and 7, and also for 11 and 13.

Problem 4:

Let Ω be a circle with center O with P, a point lying outside. Tangents from P are drawn to touch the circle at A and B. A point, T is arbitrary chosen on major arc AB, and D is the foot of T on AB. K, L, M, N are the mid points of TA, TB, TD, AB respectively. PT intersects MN at point S. Lines l_a and l_b are the reflection of OA and OB over the angle bisectors of $\angle SAL$ and $\angle SBK$, respectively. Show that l_a, l_b and TD are concurrent.

Solution 1

CLAIM: BS is the reflection of BK on the angle bisector of $\angle ABT$, which in turn is equal to the angle bisector of $\angle SBK$. Similarly, AS is the reflection of AL on the angle bisector of $\angle BAT$, which in turn is equal to the angle bisector of $\angle SAL$.



Proof. Let P' be the reflection of P on AB. Then PP' is parallel to TD, N is midpoint of PP', M is midpoint of TD, PST is collinear, and NSM is collinear. Thus, by homothethy, have $P'SD$

is also collinear and $\frac{DS}{SP'} = \frac{DT}{PP'}$.

Now, let K' be the point of intersection of BS and AP'. We shall show that $\frac{K'A}{AB} = \frac{KT}{TB}$. Since $\angle K'AB = \angle PAB = \angle KTB$, it will follow from side S-A-S criteria that $\Delta K'AB$ and ΔKTB are similar, so that $\angle ABS = \angle K'BA = \angle KBT$ as desired.

Indeed, Menalaus's theorem applied to line BSK' in $\Delta P'AD$ gives

$$1 = \frac{AB}{BD} \cdot \frac{DS}{SP'} \cdot \frac{P'K'}{K'A} = \frac{AB}{BD} \cdot \frac{DT}{PP'} \cdot \frac{P'K'}{K'A} = \frac{2AN}{BD} \cdot \frac{DT}{2PN} \cdot \frac{P'A - K'A}{K'A}$$

$$\implies 1 = \frac{AN}{PN} \cdot \frac{DT}{BD} \cdot \frac{PA - K'A}{K'A} = \frac{\tan(\angle ABT)}{\tan(\angle BAP)} \cdot \frac{PA - K'A}{K'A}$$

$$\implies 1 = \frac{AN \sin(\angle ABT) - K'A \sin(\angle ABT) \cos(\angle BAP)}{K'A \sin(\angle BAP) \cos(\angle ABT)}$$

$$\implies K'A \sin(\angle BAP + \angle ABT) = \frac{1}{2} AB \sin(\angle ABT)$$

$$\implies \frac{K'A}{AB} = \frac{\frac{1}{2}\sin(\angle ABT)}{\sin(\angle BAT)} = \frac{KT}{TB}.$$

□

Using the claim, we have that l_a and l_b are the reflection of OA and OB over the angle bisectors of $\angle BAT$ and $\angle ABT$, respectively. This implies that l_a and l_b are the altitudes of ΔABT from A and B respectively. The third altitude is TD so it follows that l_a, l_b and TD are concurrent. □

Solution 2

Lemma 1:
In a triangle ABC with D, E, F midpoints of sides BC, CA and AB respectively, let J, K, L be points on EF, FD and DE respectively, such that $AJ \perp EF, BK \perp FD$ and $CL \perp DE$.

(i) DJ, EK and FL are concurrent.

(ii) Suppose the point of concurrency in (i) is X, then AX is the A-symmedian in triangle ABC. Furthermore, this will imply that X is the concurrency point of the symmedians in triangle ABC since B or C can take the place of A. Hence, we have 6 concurrent lines. (Symmedian is the reflection of the median over the angle bisector of the respective angle)

Proof:
D, E, F are the midpoints of sides BC, CA and AB respectively, hence we have $EF \parallel BC, FD \parallel CA, DE \parallel AB$. Let AJ intersect BC at P, BK intersect CA at Q and CL intersect AB at R. Since $AJ \perp EF, BK \perp FD$ and $CL \perp DE$, we have that $AP \perp BC, BQ \perp CA$ and $CR \perp AB$ (They are altitudes in triangle ABC). It is well known that the altitudes of a triangle are concurrent, therefore, AP, BQ and CR are concurrent, hence by Ceva's theorem, we have that

$$\frac{|AR|}{|RB|} \cdot \frac{|BP|}{|PC|} \cdot \frac{|CQ|}{|QA|} = 1.$$

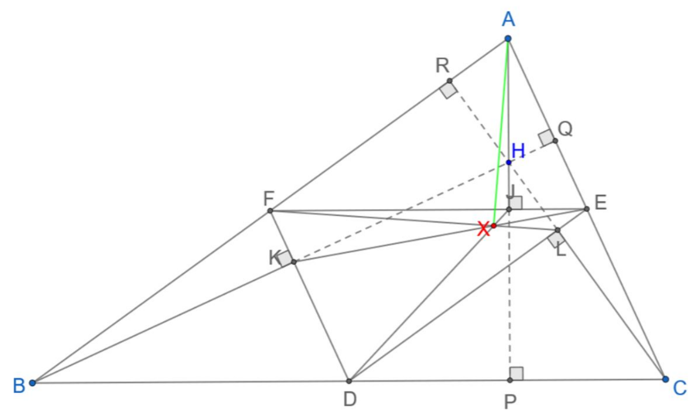
Also, from $EF \parallel BC, FD \parallel CA, DE \parallel AB$, we have the following ratio equalities:

$$\frac{|BP|}{|PC|} = \frac{|FJ|}{|JE|}, \frac{|CQ|}{|QA|} = \frac{|DK|}{|KF|} \text{ and } \frac{|AR|}{|RB|} = \frac{|EL|}{|LD|}.$$

Multiplying these ratios gives,

$$\frac{|DK|}{|KF|} \cdot \frac{|FJ|}{|JE|} \cdot \frac{|EL|}{|LD|} = \frac{|AR|}{|RB|} \cdot \frac{|BP|}{|PC|} \cdot \frac{|CQ|}{|QA|} = 1.$$

Hence, we have $\frac{|DK|}{|KF|} \cdot \frac{|FJ|}{|JE|} \cdot \frac{|EL|}{|LD|} = 1$ and by Ceva's theorem again, we have that DJ, EK and FL are concurrent. This completes the proof for (i). By Sine rule on triangle AXE, we have



$\sin(\angle XAE) = \frac{\sin(\angle AEX)}{|AX|} \cdot |EX|.$

Similarly, by Sine rule on triangle AXF, $\sin(\angle XAF) = \frac{\sin(\angle AFX)}{|AX|} \cdot |FX|.$

These two equations combine to give

$$\frac{\sin(\angle XAE)}{\sin(\angle XAF)} = \frac{|EX| \sin(\angle AEX)}{|FX| \sin(\angle AFX)} \tag{1}$$

From Sine rule on triangle DKE, $\sin(\angle DKE) = \frac{\sin(\angle EDK)}{|EK|} \cdot |DE|.$

Analogously, from triangle DLF, $\sin(\angle DLF) = \frac{\sin(\angle FDL)}{|FL|} \cdot |DF|.$

Combining the last two equations, noting that $\angle EDK = \angle FDL$, we have,

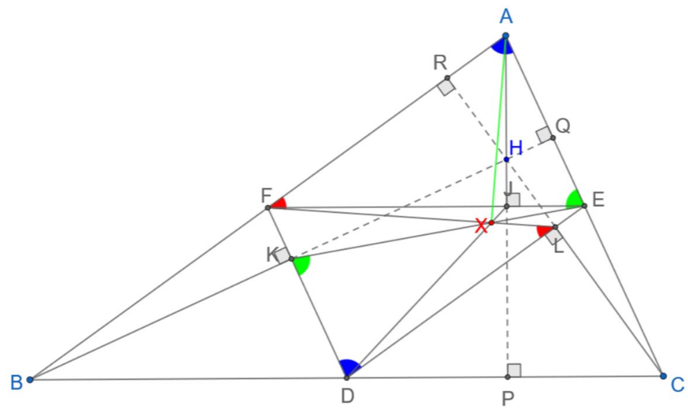
$$\frac{\sin(\angle DKE)}{\sin(\angle DLF)} = \frac{|DE||FL|}{|DF||EK|}.$$

Recall that $EF \parallel BC, FD \parallel CA, DE \parallel AB$, hence $AEDF$ is a parallelogram and $\angle AEX = \angle DKE, \angle AFX = \angle DLF$. This gives

$$\frac{\sin(\angle AEX)}{\sin(\angle AFX)} = \frac{|DE||FL|}{|DF||EK|}.$$

Hence, (1) becomes

$$\frac{\sin(\angle XAE)}{\sin(\angle XAF)} = \frac{|EX| \sin(\angle AEX)}{|FX| \sin(\angle AFX)} = \frac{|EX||DE||FL|}{|FX||DF||EK|} \tag{2}$$



We apply Extended Law of Sines on triangle ABC.
Let the circumradius of triangle ABC be R units. This gives us that $|BC| = 2R\sin A, |CA| = 2R\sin B$ and $|AB| = 2R\sin C$ where $\angle CAB = A, \angle ABC = B$ and $\angle BCA = C$.
From right angled triangle $ACR, |AR| = 2R\sin B\cos A$.
From $ED \parallel AB$ we have that $\Delta ACR \sim \Delta ECL$.
Hence $\frac{|EL|}{|AR|} = \frac{|EC|}{|AC|} = \frac{1}{2}$ (Since E is the midpoint of CA)

$$\implies |EL| = R\sin B\cos A. \tag{3}$$

Analogously, for $|FK|$, we have that from right angled triangle ABQ, $|AQ| = 2R\sin C\cos A$.
From $FD \parallel AC$ we have that $\Delta ABQ \sim \Delta FBK$.
Hence $\frac{|FK|}{|AQ|} = \frac{|FB|}{|AB|} = \frac{1}{2}$ (Since F is the midpoint of AB)

$$\implies |FK| = R\sin C\cos A. \tag{4}$$

Since $AEDF$ is a parallelogram, we have that $|AF| = |DE|$ and $|AE| = |DF$, hence $|DE| = R\sin C$ and $|DF| = R\sin B$.
Equation (2) then becomes

$$\frac{\sin(\angle XAE)}{\sin(\angle XAF)} = \frac{|EX||FL|\sin C}{|FX||EK|\sin B} \tag{5}$$

In triangle EKF, we have that $\angle KFE = \angle FEA(FD \parallel CA)\angle FEA = \angle BCA = C(EF \parallel BC)$.
Now, applying Sine rule,

$$\frac{\sin(\angle KFE)}{\sin(\angle FEK)} = \frac{|EK|}{|FK|} \implies \frac{\sin C}{\sin(\angle FEK)} = \frac{|EK|}{|FK|}.$$

Similarly, in triangle ELF, we have that $\angle LEF = \angle EFA$ ($DE \parallel AB$), $\angle EFA = \angle ABC = B$ ($EF \parallel BC$).
Now, applying Sine rule,

$$\frac{\sin(\angle LEF)}{\sin(\angle EFL)} = \frac{|FL|}{|EL|} \implies \frac{\sin B}{\sin(\angle EFL)} = \frac{|FL|}{|EL|}.$$

Combining the last two lines of equality, we have

$$\frac{|FL|}{|EK|} = \left(\frac{\sin B}{\sin C}\right)\frac{|EL|\sin(\angle FEK)}{|FK|\sin(\angle EFL)}.$$

From (3) and (4) we further get that

$$\frac{|FL|}{|EK|} = \left(\frac{\sin B}{\sin C}\right)^2 \frac{\sin(\angle FEK)}{\sin(\angle EFL)}.$$

Hence (5) gives

$$\frac{\sin(\angle XAE)}{\sin(\angle XAF)} = \left(\frac{\sin B}{\sin C}\right)\left(\frac{|EX|\sin(\angle FEK)}{|FX|\sin(\angle EFL)}\right).$$

But, from Sine rule on triangle EFX, we have that $\frac{|EX|\sin(\angle FEX)}{|FX|\sin(\angle EFX)} = 1$, but $\sin(\angle FEK) = \sin(\angle FEX)$ and $\sin(\angle EFL) = \sin(\angle EFX)$, hence $\frac{|EX|\sin(\angle FEK)}{|FX|\sin(\angle EFL)} = \frac{|EX|\sin(\angle FEX)}{|FX|\sin(\angle EFX)} = 1$.
Hence, we have

$$\frac{\sin(\angle XAE)}{\sin(\angle XAF)} = \frac{\sin B}{\sin C}. \tag{6}$$

Now, finally, we compute $\frac{\sin(\angle DAB)}{\sin(\angle DAC)}$. Applying Sine rule to triangle ABD, we have that $\sin(\angle DAB) = \frac{\sin B}{|AD|}|BD|$ and applying Sine rule to triangle ACD, we have that $\sin(\angle DAC) = \frac{\sin C}{|AD|}|CD|$. Dividing and noting that $|BD| = |CD|$ we get

$$\frac{\sin(\angle DAB)}{\sin(\angle DAC)} = \frac{\sin B}{\sin C}$$

. Hence, we have that

$$\frac{\sin(\angle XAE)}{\sin(\angle XAF)} = \frac{\sin(\angle DAB)}{\sin(\angle DAC)}.$$

But $\angle XAE + \angle XAF = \angle DAB = \angle DAC = A$. Consider the following:

$$\frac{\sin(X-a)}{\sin a} = \frac{\sin(X-b)}{\sin b}.$$

Observe that $X - a + a = X - b + b = X$

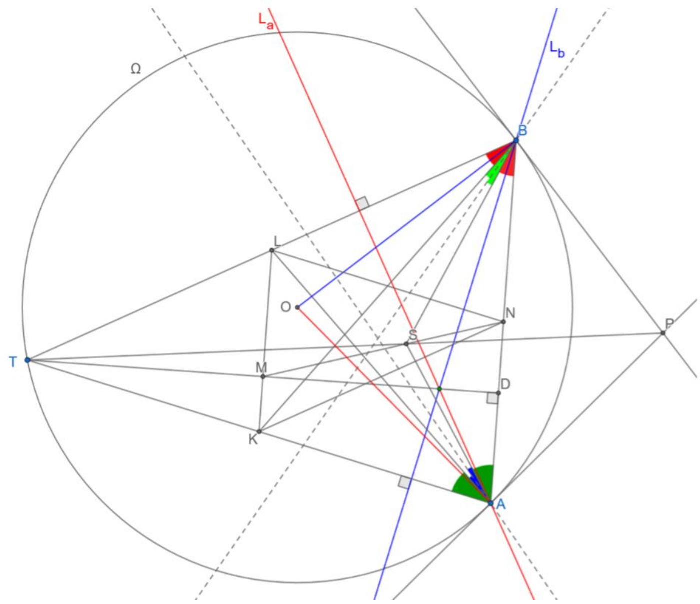
$$\frac{(\sin X\cos a - \sin a\cos X)}{\sin a} = \frac{(\sin X\cos b - \sin b\cos X)}{\sin b} \implies \sin X\cot a - \cos X = \sin X$$

$$\cot b - \cos X \implies \cot a = \cot b$$

Using this, we can conclude that $\cot(\angle XAE) = \cot(\angle DAB)$ and $\cot(\angle XAF) = \cot(\angle DAC)$ and hence, $\angle XAE = \angle DAB, \angle XAF = \angle DAC$ ($\cot(x)$ is injective in the range $(0,\pi)$). This implies that AX is the A-symmedian in triangle ABC. Similarly, BX is the B-symmedian and CX is the C-symmedian and X is the concurrency point of the symmedians in triangle ABC. This completes the proof of the lemma.

Lemma 2 Let the tangents to the circumcircle of triangle ABC at B and C meet at T. AT is the A-symmedian of triangle ABC. Proof is overlooked as this lemma is quite well known.

Now to the problem. By Lemma 2, TP is the T-symmedian of triangle ABT. Since K, L and M are midpoints of TA, TB and TD respectively, we have that K, L and M are collinear and $KL \parallel AB$. By Lemma 1, S is the point of concurrency of the symmedians in triangle ABT. Hence, AS is the reflection of AL over the angle bisector of $\angle TAB$, and therefore, the angle bisector of $\angle SAL$ is the same line as the angle bisector of $\angle TAB$. Analogously, BS is the reflection of BK



over the angle bisector of $\angle TBA$, and therefore, the angle bisector of $\angle SBK$ is the same line as the angle bisector of $\angle TBA$. Now, the reflection of OA over the angle bisector of $\angle TAB$ is the altitude from A in triangle ABT. Likewise, the reflection of OB over the angle bisector of $\angle TBA$ is the altitude from B in triangle ABT. Hence, l_a, l_b and TD are altitudes in triangle ABT and are hence concurrent. *qed*

3 Marking Scheme

Problem 1: (7 points)

Prove that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}+\sqrt{y_i}} \geq \frac{\sqrt{n}}{\sqrt{\sum_{i=1}^{n} x_i}+\sqrt{\sum_{i=1}^{n} y_i}}$$

$\forall x_i, y_i \in \mathbb{R}^+, i = 1, 2, \cdots, n$.

Full Solution:
Every full solution deserves **7 points**.

- 1 minor error. [**-1 points**]
  Or
- 2 or 3 minor errors. [**-2 points**]

Partial Solution:
Partial solutions can gain up to a maximum of **5 points**. The following are additive:

- Applying AM-HM inequality properly as in **solution 1**. [**3 points**]
- Applying QM-AM inequality properly as in **solution 1**. [**3 points**]
- Proving for the case $n = 2$ without proving **claim** as in **solution 2**. [**3 points**]
- Proving **claim** as in **solution 2**. [**5 points**]
- Checking the case $n = 1$ as in **solutions 2 & 3**. [**0 points**]
- Proving for the case $n = 2^m$ by assuming for the case $n = 2$ as in **solution 3**. [**2 points**]
- Proving for the case $n \neq 2^m$ by assuming for the case $n = 2^m$ as in **solution 3**. [**2 points**]

Good Attempts:
Good attempts that don't conform to the officials solutions can gain at most **2 points**.
Making correct claims that are vital to solving the problem may be awarded **1 point**.

Problem 2: (7 points)

Find all triples of prime numbers (p, q, r), such that $q | r - 1$, and

$$\frac{r\left(p^{q-1}-1\right)}{q^{p-1}-1}$$

is prime.

Full Solution:
Every full solution deserves **7 points**.

- 1 minor error. [**-1 points**]

Or

- 2 or 3 minor errors. [**-2 points**]

Partial Solution:
Partial solutions can gain up to a maximum of **5 points**. The following are additive:

- Observing the set of solutions (q, q, r). [**1 point**]

  Or

  Deducing the set of solutions (q, q, r) such that $r \geq 3, q|r-1$ from the case $p = q$. [**2 points**]

  Or

  Deducing the set of solutions (q, q, r) such that $r \geq 3, q|r-1$ from the case $r = s$. [**3 points**]

- Deducing the set of solutions (q, q, r) without stating $r \geq 3, q|r-1$. [**-1 point**]
- Stating the solution $(3, 2, 3)$. [**1 points**]
- Deducing $p^q - p = q^p - q, q|p-1$ by applying FLT (or otherwise), in the case $r \neq s, p \neq q$. [**1 points**]
  Note: Not points for only using FLT once.
- Showing that $(p, q) = (3, 2)$ is the only solution to $p^q - p = q^p - q, q|p-1$ in the case $q = 2$. [**2 points**]
- Showing that no solution to $p^q - p = q^p - q, q|p-1$ in the case $q \geq 3$. [**2 points**]

Good Attempts:
Good attempts that don't conform to the officials solutions can gain at most **2 points**.
Making correct claims that are vital to solving the problem may be awarded **1 point**.

Problem 3:

The numbers $2, 3, 4, \cdots, 100$ are written on a board. Chibuike and Ismail play a game of erasing numbers from the board using the following rule: If the number, a, is erased, then only numbers, b, such that $\gcd(a, b)$, can be erased in subsequent turns.
The game ends when no such b exists, to be erased, and the person that erased last wins.

If Chibuike starts the game, does there exist a winning strategy? If yes, whom? (Determine with proof.)

Full Solution:
Every full solution deserves **7 points**.

- 1 minor error. [**-1 points**]

Or

- 2 or 3 minor errors. [**-2 points**]

Good Approach:
Good approaches can gain up to a maximum of **4 points**. The following are additive:
Points are gained by showing

- If $a \in S$ and $p|a$, then $p < 100$. There are exactly 25 such primes. [**1 point**]
- If $p < 100$, then there exists exactly one $a \in S$ such that $p|a$. [**1 point**]
- If $a \in S$, then there are at most 3 primes satisfying $p|a$. Moreover, they form one of the sets $\{2, 3, 11\}$, $\{2, 3, 13\}$, or a subset of $\{2, 3, 5, 7\}$. Thus only one such number can belong to S. [**1 point**]
- If $a \in S$ has exactly 2 prime divisors, then each must have a divisor from the set $\{2, 3, 5, 7\}$. Thus, there are at most 4 such numbers. [**1 point**]
- If $a \in S$ has 3 prime divisors, then there is at most one other element of S with more than one prime divisor, except in the special cases when $S = \{66, 91, 85, \cdots\}, \{66, 91, 85, \cdots\}, \{66, 91, 95, \cdots\}, \{78, 77, 85, \cdots\}$, or $\{78, 77, 95, \cdots\}$. [**1 point**]
- If $p > 50$, then $p \in S$. [**1 point**]

Good attempts:
Good attempts that don't conform to the officials solutions can gain at most **2 points**.
Making correct claims that are vital to solving the problem may be awarded **1 point**.

Demonstrating an intuitive strategy (regardless of final answers) may be awarded up to 4 points.

Note: other solution path ways may exist so the scheme is still open for discussion depending on the need.

Problem 4:

Let Ω be a circle with center O with P, a point lying outside. Tangents from P are drawn to touch the circle at A and B. A point, T is arbitrary chosen on major arc AB, and D is the foot of T on AB. K, L, M, N are the mid points of TA, TB, TD, AB respectively. PT intersects MN at point S. Lines l_a and l_b are the reflection of OA and OB over the angle bisectors of $\angle SAL$ and $\angle SBK$, respectively. Show that l_a, l_b and TD are concurrent.

Full Solution:

Every full solution deserves **7 points**.

- 1 minor error. [**-1 points**]

Or

- 2 or 3 minor errors. [**-2 points**]

Partial Solution:

Partial solutions can gain up to a maximum of **5 points**. The following are additive:

- Applying **claim** as in **solution 1**. [**2 points**]
- Proving **claim** as in **solution 1**. [**5 points**]
  The breakdown is as follows:

  -Introducing point P'. [**1 point**]
  -Show that D, S, P' are collinear. [**1 point**]
  -Introduce point K' and apply Menalaus's theorem. [**1 point**]
  -Show that $\frac{K'A}{AB} = \frac{KT}{TB}$. [**1 point**]
  -Show that $\Delta K'AB$ and ΔKTB are similar. [**1 point**]
- Stating clearly (without proof) **Lemma 1**. [**1 point**]
- Stating **Lemma 2**. [**0 points**]
- Applying **Lemma 1** and **Lemma 2** (after claiming/stating them) to conclude. [**1 point**]
- Proving **Lemma 2** as in **solution 2**. [**1 point**]
- Proving **Lemma 1** as in **solution 2**. [**4 points**]
  The breakdown is as follows:

  -Proving part (i). [**1 point**]
  -Getting equation 5 (or something comparable). [**1 point**]
  -Getting equation 6 (after equation 5), or something comparable. [**1 point**]
  -Completing the proof. [**1 point**]

Good Attempts:

Good attempts that don't conform to the officials solutions can gain at most **2 points**.
Making correct claims that are vital to solving the problem may be awarded **1 point**.